

## Deriving the Work Done by an Inverse Square Force in Non-Calculus-Based Introductory Physics Courses

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### Abstract

I describe a method of evaluating the integral of  $\frac{1}{r^2}$  with respect to  $r$  that uses only algebra and the concept of area underneath a curve, and which does not formally employ any calculus. This is useful for algebra-based introductory physics classes (where the use of calculus is forbidden) to derive the work done by the force of one point charge or mass on another. I extend the method to evaluate any integral of the form  $\int r^n dr$  for any integer  $n \neq -1$ .

**Keywords:** Introductory physics, non-calculus, integration, inverse-square, work

### Introduction

In the United States, a large proportion [approximately 40% in 2008 according to Mulvey and Nicholson (2011)] of university students who enroll in introductory physics courses take the algebra-based version, in which formal calculus is not employed. The popularity of the algebra-based courses is in part due to typical medical school requirements and the Medical College Admissions Tests syllabus (American Association of Medical Colleges (2008)), for which algebra-based version of the introductory courses satisfies the physics requirement.

In algebra-based introductory physics courses, the result for the work done by an inverse square force, such as the Coulomb or gravitational force between point particles, is usually given without proof, since the rigorous derivation of the result requires knowledge of integral calculus. Here, I describe a way that the work can be obtained without any formal calculus, based on work being the area under the force vs. displacement curve.

### The derivation for area under a $1/r^2$ curve

The concept of work in algebra-based introductory physics courses is typically introduced for the special case of a constant force, in which case the work done is  $W = Fd$ , where  $F$  is the component of the constant force in the direction of the displacement and  $d$  is the distance travelled. This corresponds to the area enclosed in the rectangle created by the constant force vs. displacement curve. The work done by a variable force is later explained as the area underneath the force vs. displacement curve, typically with the aid of a figure such as shown in Fig. 1, where the area underneath the curve has been divided into many small rectangular blocks. In the case of a Hooke's law force which is proportional to the displacement,  $F = kx$ , the area underneath the curve between  $x_i$  and  $x_f$  is  $W = \frac{1}{2}k(x_f^2 - x_i^2)$ . This can be obtained without calculus because it is the area of a right-angled trapezoid, which is the sum of the area of a rectangle and a right-angled

triangle. However, in the case work done by forces that vary as the inverse-square of the distance,  $F = k/r^2$ , the work done in moving a particle from  $r_i$  to  $r_f$ ,

$$W = k \int_{r_i}^{r_f} dr \frac{1}{r^2} = k \left( \frac{1}{r_i} - \frac{1}{r_f} \right) \quad (1)$$

is usually simply stated as a result whose derivation requires knowledge of mathematics that is beyond the scope of a non-calculus based course. Consequently, students often do not remember this result. In my experience, when students are asked in tests for the work done by a force  $F = k/r^2$  in moving an object from a distance  $R$  to  $R+d$ , instead of the correct answer  $W = k \left( \frac{1}{R} - \frac{1}{R+d} \right)$ , a significant fraction of the students recall the constant force result and answer  $W = Fd$ , where they typically use the initial force  $k/R^2$  for  $F$ . This error is less common when a similar question is asked for a Hooke's law force, in my opinion, because students are explicitly shown that the work done for a spring is that of the area underneath the curve, whereas for the inverse-square forces, they are simply told the result as a fact. If this is the case, the solution is to derive the result for them in the inverse-square force case. But how do we do this without the use of integral calculus, and in a simple and straightforward way?

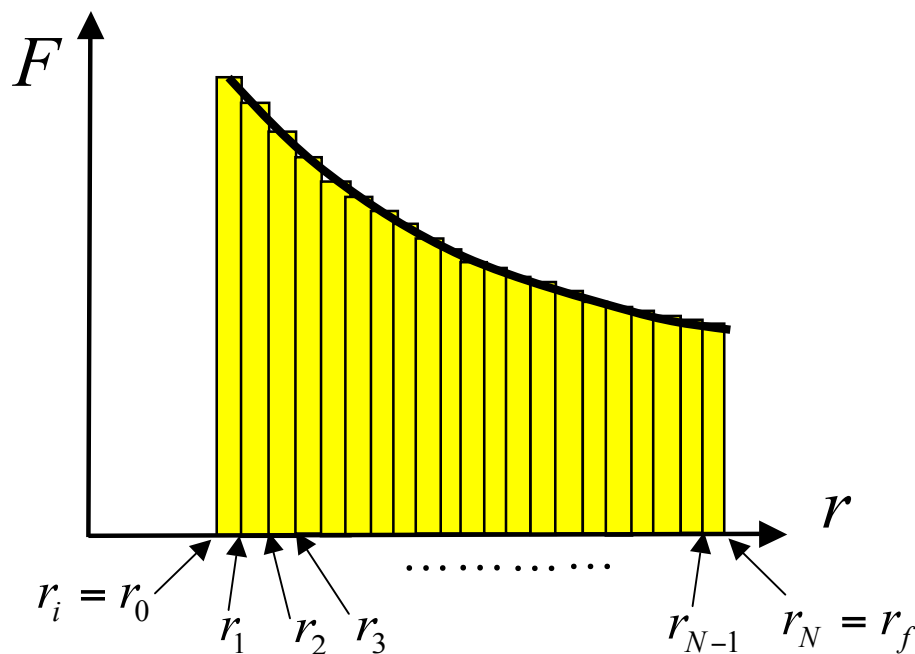


Figure 1: Area underneath a curve  $F$  from  $r_i$  to  $r_f$ , divided into  $N$  subintervals, where the limits of sub-intervals are  $r_i = r_0, r_1, r_2, \dots, r_{N-1}, r_N = r_f$ . The area  $A_j$  under the curve in the sub-interval from  $r_j$  and  $r_{j-1}$  is equal to  $(r_j - r_{j-1}) \langle F \rangle$ , where  $\langle F \rangle$  is the average of the function  $F$  over that interval.

The integral  $\int_{r_i}^{r_f} dr \frac{1}{r^2}$  is the area underneath the  $\frac{1}{r^2}$  curve from  $r_i$  to  $r_f$ . Divide the interval  $r_i$  to  $r_f$  into  $N$  equally spaced sub-intervals, where  $N$  is a large number and the  $j$ -th sub-interval is from  $r_{j-1}$  to  $r_j$  (with  $r_0 = r_i$  and  $r_N = r_f$ ), as shown in Fig. 1. The area under the curve in the  $j$ -th sub-interval from  $r_{j-1}$  to  $r_j$  is

$$A_j = (r_j - r_{j-1}) \left\langle \frac{1}{r^2} \right\rangle \tag{2}$$

where  $\left\langle \frac{1}{r^2} \right\rangle$  is the average of  $\frac{1}{r^2}$  over the sub-interval. Since the number of intervals  $N$  is large, the difference between  $r_{j-1}$  and  $r_j$  is very small and we can substitute any  $r$  from the interval  $r_{j-1}$  and  $r_j$  into the  $r$  in  $\left\langle \frac{1}{r^2} \right\rangle$  and obtain an accurate result. Let us choose to substitute (with the benefit of hindsight!) the geometric mean of the lower and upper limits of the sub-interval,  $r \rightarrow (r_j r_{j-1})^{1/2}$ . Thus,  $\left\langle \frac{1}{r^2} \right\rangle \rightarrow \frac{1}{r_j r_{j-1}}$ , which gives

$$A_j = \frac{r_j - r_{j-1}}{r_j r_{j-1}} = \frac{1}{r_{j-1}} - \frac{1}{r_j} \tag{3}$$

The area underneath the  $\frac{1}{r^2}$  curve from  $r_i$  to  $r_f$  is the sum of the areas underneath the curves in the sub-intervals  $r_0 (= r_i)$  to  $r_1$ ,  $r_1$  to  $r_2$ ,  $r_2$  to  $r_3$ , and so on until  $r_{N-1}$  to  $r_N (= r_f)$ , which gives

$$\begin{aligned} \int_{r_i}^{r_f} dr \frac{1}{r^2} &= \sum_{j=1}^N A_j = \sum_{j=1}^N \left( \frac{1}{r_{j-1}} - \frac{1}{r_j} \right) = \left( \frac{1}{r_0} - \frac{1}{r_1} \right) + \left( \frac{1}{r_1} - \frac{1}{r_2} \right) + \dots + \left( \frac{1}{r_{N-2}} - \frac{1}{r_{N-1}} \right) + \left( \frac{1}{r_{N-1}} - \frac{1}{r_N} \right) \\ &= \frac{1}{r_0} - \frac{1}{r_N} = \frac{1}{r_i} - \frac{1}{r_f} \end{aligned} \tag{4}$$

where the second-to-last equality results from second term in the first parenthesis cancelling the first term in the second parenthesis, the second term in the second parenthesis cancelling the first term in the third parenthesis, and so on.

**Generalization to  $\int r^n dr$  for integers  $n \neq 1$**

The above result can be generalized to integrate  $r^n$  for any integer  $n$ , except  $n = -1$ . As in the case of integrating  $\frac{1}{r^2}$ , one divides the integration interval from  $r_i$  to  $r_f$  into sub-intervals given

by  $r_0 = r_i; r_1; r_2; \dots; r_{N-1}; r_N = r_f$ , where  $N$  is a large number, so that the sub-intervals are very small. I deal with the cases  $\frac{1}{r^{n+1}}$  and  $r^n$  for  $n \geq 1$  separately. (The  $r^0$  case is trivial because  $r^0 = 1$ .)

Evaluating  $\int \frac{1}{r^{n+1}} dr$  for integers  $n \geq 1$

The area under the curve  $\frac{1}{r^{n+1}}$  in the sub-interval from  $r_{j-1}$  to  $r_j$  is

$$A_j = (r_j - r_{j-1}) \left\langle \frac{1}{r^{n+1}} \right\rangle \tag{5}$$

where  $\left\langle 1/r^{n+1} \right\rangle$  is the average over the interval. Because the sub-interval is so small, we can replace  $1/r^{n+1}$  with any function of  $r_{j-1}$  and  $r_j$ , which gives  $1/r^{n+1}$  when both  $r_{j-1}$  and  $r_j$  are replaced by  $r$ . Again, with the benefit of hindsight, we choose

$$\left\langle \frac{1}{r^{n+1}} \right\rangle = \frac{1}{n} \frac{r_j^{n-1} + r_j^{n-2}r_{j-1} + r_j^{n-3}r_{j-1}^2 + \dots + r_j r_{j-1}^{n-2} + r_{j-1}^{n-1}}{r_j^n r_{j-1}^n}. \tag{6}$$

(Note that there are  $n$  terms in the numerator on the right hand side of Eq. (6), so when  $r_j$  and  $r_{j-1}$  are replaced by  $r$ , the numerator becomes  $nr^{n-1}$ .) With this choice,

$$A_j = (r_j - r_{j-1}) \left\langle \frac{1}{r^{n+1}} \right\rangle = \frac{(r_j - r_{j-1})}{n} \frac{(r_j^{n-1} + r_j^{n-2}r_{j-1} + r_j^{n-3}r_{j-1}^2 + \dots + r_j r_{j-1}^{n-2} + r_{j-1}^{n-1})}{r_j^n r_{j-1}^n}. \tag{7}$$

The numerator in Eq. (7) “telescopes”; i.e.,

$$(r_j - r_{j-1})(r_j^{n-1} + r_j^{n-2}r_{j-1} + r_j^{n-3}r_{j-1}^2 + \dots + r_j r_{j-1}^{n-2} + r_{j-1}^{n-1}) = r_j^n + r_j^{n-1}r_{j-1} + r_j^{n-2}r_{j-1}^2 + \dots + r_j r_{j-1}^{n-1} - (r_j^{n-1}r_{j-1} + r_j^{n-2}r_{j-1}^2 + \dots + r_j r_{j-1}^{n-1} + r_{j-1}^n) = r_j^n - r_{j-1}^n \tag{8}$$

and Eq. (7) becomes

$$A_j = \frac{1}{n} \frac{(r_j^n - r_{j-1}^n)}{r_j^n r_{j-1}^n} = \frac{1}{n} \left( \frac{1}{r_{j-1}^n} - \frac{1}{r_j^n} \right). \tag{9}$$

Therefore,

$$\begin{aligned} \int_{r_i}^{r_f} dr \frac{1}{r^{n+1}} &= \sum_{j=1}^N A_j = \frac{1}{n} \sum_{j=1}^N \frac{1}{r_{j-1}^n} - \frac{1}{r_j^n} = \frac{1}{n} \left[ \left( \frac{1}{r_0^n} - \frac{1}{r_1^n} \right) + \left( \frac{1}{r_1^n} - \frac{1}{r_2^n} \right) + \dots + \left( \frac{1}{r_{N-2}^n} - \frac{1}{r_{N-1}^n} \right) + \left( \frac{1}{r_{N-1}^n} - \frac{1}{r_N^n} \right) \right] \\ &= \frac{1}{n} \left[ \frac{1}{r_0^n} - \frac{1}{r_N^n} \right] = \frac{1}{n} \left[ \frac{1}{r_i^n} - \frac{1}{r_f^n} \right]. \end{aligned} \tag{10}$$

### Evaluating $\int r^n dr$ for integers $n \geq 1$

The area under the curve  $r^n$  in the sub-interval from  $r_{j-1}$  to  $r_j$  is  $A_j = (r_j - r_{j-1})\langle r^n \rangle$ . With the benefit of hindsight, we replace

$$\langle r^n \rangle = \frac{1}{n+1} (r_j^n + r_j^{n-1}r_{j-1} + r_j^{n-2}r_{j-1}^2 + \dots + r_j r_{j-1}^{n-1} + r_{j-1}^n) \quad (12)$$

(Note that there are  $n+1$  terms in the parentheses of the above equation, which gives  $(n+1)r^n$  when  $r_j$  and  $r_{j-1}$  are replaced by  $r$ .) Using the “telescoping” result, Eq. (8), gives

$$A_j = (r_j - r_{j-1})\langle r^n \rangle = \frac{(r_j - r_{j-1})}{n+1} (r_j^n + r_j^{n-1}r_{j-1} + r_j^{n-2}r_{j-1}^2 + \dots + r_j r_{j-1}^{n-1} + r_{j-1}^n) = \frac{1}{n+1} (r_j^{n+1} - r_{j-1}^{n+1}), \quad (13)$$

and therefore

$$\begin{aligned} \int_{r_i}^{r_f} dr r^n &= \sum_{j=1}^N A_j = \frac{1}{n+1} \sum_{j=1}^N r_j^{n+1} - r_{j-1}^{n+1} = \frac{1}{n+1} [(r_1^{n+1} - r_0^{n+1}) + (r_2^{n+1} - r_1^{n+1}) + \dots + (r_{N-1}^{n+1} - r_{N-2}^{n+1}) + (r_N^{n+1} - r_{N-1}^{n+1})] \\ &= \frac{1}{n+1} [r_N^{n+1} - r_0^{n+1}] = \frac{1}{n+1} [r_f^{n+1} - r_i^{n+1}]. \end{aligned} \quad (14)$$

### Conclusion

In conclusion, for the special but important case of the work of an inverse-square force  $F = k/r^2$ , one can demonstrate that the work done by the force in moving a particle from  $r_i$  to  $r_f$  is  $k(\frac{1}{r_i} - \frac{1}{r_f})$  without the formal use of integral calculus. Many students in non-calculus-based introductory physics classes have a fixation with the constant force equation for work,  $W = Fd$ , and will use it even when the force is not constant. Therefore, an explicit demonstration of the result for the inverse-square force case should be helpful in preventing these students from making that common mistake. The method described is also extended to evaluate the integral of  $r^n$  with respect to  $r$  for any integer  $n \neq 1$ .

### References

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